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CONTACT MANIFOLDS

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1. Introduction

This paper is a study of the differential topology of contact manifolds. In § 2, the known results on contact manifolds will be reviewed. The problem of existence of a global contact form will be studied, straightening out a minor error in Gray [4]. The reduction of the structure group of the tangent bundle will be examined. In § 3, the Kervaire semi-characteristic will be defined and studied. Classically, there are two types of contact manifolds: a) the bundle of tangent corays to a closed manifold and b) the spheres. The semi-characteristic distinguishes these types. In § 4, the semi-characteristic will be placed in a cobordism framework. This provides insight into just what the invariant measures. In § 5, it will be shown that a closed oriented contact manifold of dimension $8k + 5$ is the boundary of a compact almost complex manifold. In § 6, the characteristic classes of contact manifolds will be studied.

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2. Review of contact manifolds

Let $(x^1, \cdots, x^n, y^1, \cdots, y^n, z)$ be coordinates in Euclidean $(2n + 1)$-space R^{2n+1}, and let α_0 be the 1-form on R^{2n+1} defined by $\alpha_0 = dz - \sum y^i dx^i$. This form is completely characterized by the fact that $\alpha_0 \wedge (d\alpha_0)^n \neq 0$ in the sense that any form with this property has the given expression in suitably chosen local coordinates. A diffeomorphism $f\colon U \to V$ between open sets of R^{2n+1} is called a *contact transformation* if $f^*\alpha_0 = \rho\alpha_0$ for some nonzero real function ρ on U. The collection Γ of all contact transformations forms a pseudogroup.

The systematic study of pseudogroups began with the work of Sophus Lie on transformation groups [9], and volume two of his work is devoted to the study of contact transformations.

An odd dimensional manifold M^{2n+1} is called a *contact manifold* if there is an open cover $\{U_i\}$ of M with homeomorphisms $f_i\colon U_i \to V_i \subset R^{2n+1}$ so that when $f_i \cdot f_j^{-1}$ is defined, $f_i \circ f_j^{-1} \in \Gamma$. Two such systems $\{U_i, f_i\}, \{U'_i, f'_i\}$ are equi-

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valent if $f'_i \circ f_i^{-1} \in \Gamma$ whenever defined, and a *contact structure* on M is an equivalence class of such coordinate systems.

Being given a contact structure on M determines a subline bundle ξ of the cotangent bundle τ^* of M. Specifically, if $x \in M$ and (U_i, f_i) is a chart at x compatible with the contact structure, then the fiber of ξ at x is the subspace spanned by $f_i^*(\alpha_0)$. To see that this is well-defined, one needs only to note that on $U_i \cap U_j$, $f_i^*(\alpha_0) = \rho_{ij} f_j^*(\alpha_0)$ for some nonvanishing function ρ_{ij} on $U_i \cap U_j$.

The contact distribution on M is the sub-bundle of the tangent bundle of M described as follows. It $x \in M$, and (U, f) is a chart at x compatible with the contact structure, then the fiber of the contact distribution at x consists of those tangent vectors at x annihilated by $f^*(\alpha_0)$. In other words the contact distribution is the annihilator of the contact line bundle under the dual pairing of tangent and cotangent bundles.

Proposition 2.1. *If M is a $(2n+1)$-dimensional contact manifold, the determinant bundle of the cotangent bundle $\det \tau^*$ of M is the $(n+1)$-st tensor power of the contact line bundle ξ. In particular,*

a) *if n is odd, M is oriented;*

b) *if n is even, ξ is isomorphic to* $\det \tau^*$ *and the contact distribution is oriented.*

Proof. Being given a chart (U_i, f_i) compatible with the contact structure of M, $\omega_i = f_i^*(\alpha_0) \wedge (df_i^*(\alpha_0))^n$ is a nonvanishing section of $\det \tau^*$ over U_i. For a second chart (U_j, f_j), $f_i^*(\alpha_0) = \rho_{ij} f_j^*(\alpha_0)$ gives $\omega_i = \rho_{ij}^{n+1}\omega_j$. This is the relationship of coordinate transformation between a line bundle and its $(n+1)$-st tensor power. For any line bundle η, $\eta \otimes \eta = \eta^2$ is trivial, and in fact has a preferred class of trivializations given by the positive cone of all $e \otimes e$ with $e \neq 0$. Thus if n is odd, ξ^{n+1} has a preferred trivialization, and if n is even there is a preferred isomorphism of ξ and ξ^{n+1}, which gives a) and b).

Remark. This is basically Proposition 2.2.1 of Gray [4]. There is an error in the last six lines of the proof of Proposition 2.2.1 which invalidates part (ii) of the result for n odd. Specifically, there is a globally defined vector field v which is complementary to the $2n$-distribution of hyperplanes defined by $\alpha_i = 0$ if and only if the distribution consists of *oriented* hyperplanes. Gray gives a counterexample to his own result (Example 4 of § 2.4): Let $M^{2n+1} = R^{n+1} \times P^n(R)$, where $P^n(R)$ denotes n-dimensional real projective space. Let $(x^0, \cdots, x^n)$ be coordinates in R^{n+1}, and $(t_0, \cdots, t_n)$ homogeneous coordinates in $P^n(R)$. Let $U_i \subset R^{n+1} \times P^n(R)$ be defined by $t_i \neq 0$ and $\alpha_i = t_i^{-1}(\sum t_j dx^j)$. Then the α_i define a contact structure on M for which $\rho_{ij} = t_j/t_i$, and ξ is the nontrivial line bundle over M induced from the canonical line bundle over $P^n(R)$. When n is odd, $P^n(R)$ is orientable and $\det \tau^*$ is trivial.

This example may be globalized. For any manifold N^{n+1}, the projective space bundle $M^{2n+1} = RP(\tau^*)$ of the cotangent bundle of N is a contact manifold. Specifically, given a chart U of N with local coordinates $(x_0, \cdots, x_n)$, one has a chart $U \times R^{n+1}$ for the cotangent bundle with local coordinates $(x_0, \cdots, x_n,$

$q_0, \cdots, q_n)$ where a form is expressed as $\sum q_i dx_i$ in these coordinates. The q_i may be considered as homogeneous coordinates in M. Letting U_i be the open set in M lying over U in N and defined by $q_i \neq 0$, we see that $\alpha_i = q_i^{-1}(\sum q_j dx_j)$ is a form on U_i. On U_i an explicit coordinate function $f\colon U_i \to R^{2n+1}$ with $f^*(\alpha_0) = \alpha_i$ is given by $f(x_0, \cdots, x_n, q_0, \cdots, q_n) = (x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, -q_0/q_i, \cdots, -q_{i-1}/q_i, -q_{i+1}/q_i, \cdots, -q_n/q_i, x_i)$. If U' is another chart in N with local coordinates $(x_0', \cdots, x_n')$, then on $(U \times R^{n+1}) \cap (U' \times R^{n+1})$ the coordinates are related by $x_i' = \phi_i(x_0, \cdots, x_n), q_i' = \sum q_j(\partial x_j/\partial x_i')$, so that $\sum q_i dx_i = \sum q_i' dx_i'$ on th ecotangent bundle. Then $\alpha_j' = q_j'^{-1}(\sum q_k' dx_k') = (q_i/q_j')\alpha_i$ on $U_i \cap U_j'$.

Proposition 2.2. *The structural group of the tangent bundle of a* $(2n + 1)$-*dimensional contact manifold reduces to the subgroup* $C_{2n+1} \subset 0(2n + 1)$ *generated by* $1 \times U(n)$ *and* $-1 \times$ *(conjugation), where* R^{2n+1} *is considered as* $R \times C^n$.

Proof. Choose a Riemannian metric on M, reducing the structural group to $0(2n + 1)$. The existence of the distribution of hyperplanes $\alpha_i = 0$ gives a further reduction to $0(1) \times 0(2n)$. For each point x of M, one chooses a chart (U_i, f_i) compatible with the contact structure of M, and defines a 1-form α_x on U_i by $f_i^*(\alpha_0)/f_i^*(\alpha_0)(v_i)$ where v_i is the unit vector field complementary to the distribution $\alpha_i = 0$ with $f_i^*(\alpha_0)(v_i) > 0$. (In effect this chooses the 1-form of length 1 in ξ with the Riemannian metric induced on τ^*.) $(f_i^{-1*})(\alpha_*)$ is then a 1-form on the open set $f_i(U_i) \subset R^{2n+1}$, and since it satisfies the condition $\alpha \wedge (d\alpha)^n \neq 0$, one may find an open set $W_x \subset f_i(U_i)$ containing $f_i(x)$ and a diffeomorphism $g_x\colon W_x \to R^{2n+1}$ onto an open subset for which $g_x^*(\alpha_0) = (f_i^{-1})^*(\alpha_x)$. Let $U_x \subset f_i^{-1}(W_x)$ be the component of $f_i^{-1}(W_x)$ containing x. Then $f_x = g_i \circ f_i$ gives a chart (U_x, f_x) near x compatible with the given contact structure on M and on which $\alpha_x = f_x^*(\alpha_0)$ is a unit field in ξ.

Because U_x is connected, there are only two possible one-forms α_x on U_x with the given property. If $f_x\colon U_x \to R^{2n+1}$ with $f_x^*(\alpha_0) = \alpha_x$, and $\phi\colon R^{2n+1} \to R^{2n+1}$ is given by $\phi(x^1, \cdots, x^n, y^1, \cdots, y^n, z) = (x^1, \cdots, x^n, -y^1, \cdots, -y^n, -z)$, then $\phi \circ f_x\colon U_x \to R^{2n+1}$ gives $(\phi \circ f_x)^*(\alpha_0) = -\alpha_x$ which is the other form.

One may then consider M as having the contact structure defined by a collection of open sets (U_i, f_i) with U_i connected and $f_i^*(\alpha_0) = \alpha_i$ a "unit" 1-form.

On each open set U_i, one may then write $d\alpha_i|_{\alpha_i=0} = \sum_1^n \sigma_i^k \wedge \sigma_i^{n+k}$, and let $g_{ij}\colon U_i \cap U_j \to 0(1) \times 0(2n)$ represent the tangent bundle of M in the covering $\{U_i\}$. Being given a point x on $U_i \cap U_j$ with $\alpha_i(x) = \alpha_j(x)$ the bundles ξ and det τ^* over U_i and U_j have the same orientation (as imparted by α_i and α_j) at x. Thus $g_{ij}(x)$ lies in $SO(1) \times SO(2n)$, and if E is the matrix of coefficients of the form $\sum \sigma_j^k \wedge \sigma_j^{n+k}$, then $g_{ij}E = Eg_{ij}$ so that $g_{ij}(x) \in 1 \times U(n)$. If on the other hand $\alpha_j(x) = -\alpha_i(x)$, then replacing (U_j, f_j) by $(U_j, f_j \circ \phi)$ changes g_{ij} to $g_{ij} \circ (-1 \times \text{conjugation})$ which belongs to $1 \times U(n)$. Thus $g_{ij}(x)$ belongs to C_{2n+1}.

Notes. a) This is in essence contained in the proof of Theorem 2.3.2 of Gray [4], who attributes it to Chern [2].

b) It shold be noted that $-1 \times$ conjugation is in the normalizer of $1 \times U(u)$ so that C_{2n+1} contains $1 \times U(n)$ as a subgroup of index 2. If n is odd, then $C_{2n+1} \subset SO(2n+1)$ while for n even, $C_{2n+1} \cap SO(2n+1) = 1 \times U(n)$, and C_{2n+1} is contained in $0(1) \times SO(2n)$. This repeats the assertions of Proposition 2.1.

c) The author apologizes slightly for the cumbersomeness of this proof. This is due in part to his lack of understanding of forms.

d) The reduction depends only on the choice of a Riemannian metric.

e) For $M^{2n+1} = RP(\tau^*)$, where τ^* is the cotangent bundle of N^{n+1}, let $\pi : M^{2n+1} \rightarrow N^{n+1}$ be the bundle projection. Considering M as lines in the fibers of τ^*, let λ be the line bundle over M given by pairs consisting of a line in a fiber of τ^* and a point on that line which is the contact line bundle. Choosing a Riemannian metric on N to identify τ and τ^*, one has $\theta \otimes 1 = \pi^*(\tau^*) \otimes \lambda$ where θ is the bundle along the fibers of π, so that the tangent (or cotangent) bundle of M is given by $\lambda \oplus \theta \oplus (\lambda \otimes \theta)$. The structure group of this bundle is generated by $1 \times 0(n)$ and $-1 \times$ conjugation, where $0(n) \subset U(n)$ by complexification.

Now one would like to understand the existence of a global 1-form on M defining the contact structure. For convenience, make

Definition 2.3. A *contact form* on a manifold M of dimension $2n+1$ is a 1-form α such that $\alpha \wedge (d\alpha)^n$ is never zero. Two contact forms α and α' on M are *strictly equivalent* if there is a positive function $\rho > 0$ on M with $\alpha' = \rho\alpha$. A manifold M with a chosen strict equivalence class of contact forms will be called a *strict contact manifold*.

It is proper to refer to M as a strict contact manifold since, if α is a contact form on M, M admits a contact structure given by those charts $\{U, f\}$ for which $f^*(\alpha_0) = \rho\alpha$ for some nonvanishing function ρ on U. In fact a strict contact manifold M is precisely a contact manifold M together with a homotopy class of trivializations of the contact line bundle ξ. This point is thoroughly discussed by Martinet [10].

If M is a $(2n+1)$-dimensional contact manifold with n even, then by Proposition 2.1, b), the choice of a trivialization of ξ or of det τ^* is equivalent, and thus a strict contact manifold of dimension $4k+1$ is just an oriented contact manifold.

If M is a $(2n+1)$-dimensional contact manifold with n odd, then M is an oriented manifold automatically. The choice of a trivialization of ξ is the choice of a trivializaton of the complement of the contact distribution, or a choice of orientation on the contact distribution.

Note. If n is odd, M is oriented, while if n is even the contact distribution is oriented. A strict contact structure is the choice of an orientation on the other one.

From Proposition 2.2, one sees that the chioce of a Riemannian metric on a strict contact manifold M of dimension $2n + 1$ gives a reduction of the tangent bundle of M to $1 \times U(n)$. Alternately, if M is a Riemannian contact manifold of dimension $2n + 1$, the choice of a strict equivalence class of contact forms on M is a reduction of the structural group of the tangent bundle from C_{2n+1} to $1 \times U(n)$.

In Martinet [10] it is shown that every compact orientable 3-dimensional manifold possesses a strict contact structure. For 1-dimensional manifolds, this is also true, since S^1 has a strict contact structure.

It should be noted that this is best possible; specifically, we have

Proposition 2.4. *For $n \geq 2$ there is a closed oriented connected manifold of dimension $2n + 1$ with no contact structure.*

One such manifold is $SU(3)/SO(3)$ for $n = 2$, and $(SU(3)/SO(3)) \times S^{2n-4}$ for $n > 2$. The manifold $SU(3)/SO(3)$, first noted by Calabi, is a simply connected 5 dimensional manifold with the Stiefel-Whitney number $w_2w_3[SU(3)/SO(3)] \neq 0$.

Letting M^{2n+1} be one of the given manifolds, we see that M is connected compact and simply connected, and $w_3(M) \neq 0$. If M admits a contact structure, it admits a strict contact structure, since $H^1(M; Z_2) = 0$. Then the tangent bundle of M reduces to $1 \times U(n)$, and $w_3(M)$ must be zero, which is a contradiction.

This is hardly exciting; there are many examples. It seems desirable to give some other types of examples.

Let N^{n+1} be a closed $(n + 1)$-dimensional manifold and $M^{2n+1} = RP(\tau^*)$, the projective space bundle of the cotangent bundle.

The mod 2 cohomology of $RP(\tau^*)$ is well-known. If $c \in H^1(RP(\tau^*); Z_2)$ is the Stiefel-Whitney class of the double cover of $RP(\tau^*)$ by the sphere bundle, and $\pi\colon RP(\tau^*) \to N$ is the bundle projection, then $H^*(RP(\tau^*); Z_2)$ is the free $H^*(N; Z_2)$ module via π^* on $1, c, \cdots, c^n$ and $c^{n+1} = \pi^*(w_1(N))c^n + \cdots + \pi^*(w_r(N))c^{n+1-r} + \cdots + \pi^*(w_{n+1}(N))$. The Stiefel-Whitney class of $RP(\tau^*)$ is

$$\begin{aligned} &\{1 + w_1(N) + \cdots + w_{n+1}(N)\} \\ &\quad \cdot\{(1 + c)^{n+1} + w_1(N)(1 + c)^n + \cdots \\ &\qquad + w_r(N)(1 + c)^{n+1-r} + \cdots + w_{n+1}(N)\}\,, \end{aligned}$$

where π^* has been deleted from the notation.

One easily has $w_1(RP(\tau^*)) = (n + 1)c$, so for n even and $n \geq 2$, $RP(\tau^*)$ is nonorientable and the contact structure is not a strict contact structure. Then supposing n is odd and $n \geq 3$, we have

$$\begin{aligned} w_3(RP(\tau^*)) = {}& w_3(N) + w_2(N)\{(n + 1)c + w_1(N)\} \\ & + w_1(N)\left\{\binom{n+1}{2}c^2 + nw_1(N)c + w_2(N)\right\} \end{aligned}$$

$$+\binom{n+1}{3}c^3 + w_1(N)\binom{n}{2}c^2 + w_2(N)\binom{n-1}{1}c + w_3(N)$$
$$= w_3(N) + w_2(N)w_1(N) + \tfrac{1}{2}(n+1)c^2w_1(N) + w_1^2(N)c$$
$$+ w_1(N)w_2(N) + \tfrac{1}{2}(n-1)c^2w_1(N) + w_3(N)$$
$$= w_1^2(N)c + c^2w_1(N) .$$

Thus, if N is any nonorientable $(n+1)$-manifold, e.g., $RP(2) \times S^{n-1}$, then $w_3(RP(\tau^*)) \neq 0$. If the tangent bundle of $RP(\tau^*)$ was reducible to $1 \times U(n)$, the third Stiefel-Whitney class would be zero. Thus one has:

Proposition 2.5. *If $n \geq 2$, there is a closed connected contact manifold of dimension $2n+1$, which is not a strict contact manifold.*

It should be noted that Martinet's result says this cannot happen in dimension 3, and since every one dimensional manifold is orientable, it cannot happen in dimension one.

3. The semi-characteristic

Following Kervaire [6], one defines the semi-characteristic $s\chi(M) \in Z_2$ for an odd dimensional closed manifold M^{2n+1} to be the mod 2 reduction of

$$\sum_{i=0}^{n} (-1)^i \dim H^i(M; Z_2) ,$$

where dim denotes the dimension of the Z_2 vector space. For an even dimensional closed manifold M^{2n} of even Euler characteristic one lets $s\chi(M) = \sum_{i=0}^{n-1} (-1)^i \dim H^i(M; Z_2) + (-1)^n \frac{1}{2} \dim H^n(M; Z_2)$ which is one-half of the Euler characteristic.

Classically, there are two types of contact manifolds: a) the bundle of tangent corays of a manifold, and b) the spheres. The semi-characteristic distinguishes these types (that the sphere does not have type a) was noted by Boothby and Wang [1]).

Proposition 3.1. *The semi-characteristic of the sphere S^{2n+1} is nonzero, and if M^{2n+1} is the sphere bundle of the cotangent bundle of a closed manifold N^{n+1}, then the semi-characteristic of M is zero.*

Proof. $s\chi(S^{2n+1}) = \dim H^0(S^{2n+1}; Z_2) = 1$. If M^{2n+1} is the sphere bundle of the cotangent bundle of a closed connected manifold N^{n+1}, one has the exact Gysin sequence

$$\cdots \longrightarrow H^{i-(n+1)}(N; Z_2) \xrightarrow{\beta} H^i(N; Z_2)$$
$$\xrightarrow{\pi^*} H^i(M; Z_2) \longrightarrow H^{i-n}(N; Z_2) \xrightarrow{\beta} \cdots$$

with β being multiplication by $w_{n+1}(\tau^*)$. For $0 \leq i < n$, this gives $\pi^*: H^i(N; Z_2) \xrightarrow{\cong} H^i(M; Z_2)$, and the nontrivial part of the sequence is

$$0 \longrightarrow H^n(N) \longrightarrow H^n(M) \longrightarrow H^0(N)$$
$$\xrightarrow{\beta} H^{n+1}(N) \longrightarrow H^{n+1}(M) \longrightarrow H^1(N) \longrightarrow 0 .$$

Since N is connected, $H^0(N) \cong H^{n+1}(N) \cong Z_2$, and β is nontrivial if and only if $w_{n+1}(\tau^*) \neq 0$. Now $\langle w_{n+1}(\tau^*), [N]\rangle$ is the mod 2 reduction of the Euler characteristic of $N, \chi(N)$. If $\chi(N) \equiv 0 \pmod 2$, then $s\chi(M) \equiv \sum_{i=0}^{n} \dim H^i(N) + \dim H^0(N) \equiv \sum_{i=0}^{n+1} \dim H^i(N) \equiv \chi(N) \equiv 0$, while if $\chi(N) \equiv 1 \pmod 2$, then $s\chi(M) \equiv \sum_{i=0}^{n} \dim H^i(N) \equiv \chi(N) - 1 \equiv 0$. q.e.d.

Now recall that a manifold with or without boundary V is said to be *k-parallellizeable* if the restriction of the tangent bundle of V to the k-skeleton of V is trivial. (Alternatively, V is k-parallellizeable if for every finite complex X of dimension less than or equal to k and every map $f: X \to V$, $f^*(\tau_v)$ is trivial.)

For any integer k, one lets $\phi(0, k)$ be the number of integers s with $0 < s \leq k$, which are congruent to $0, 1, 2$, or 4 modulo 8.

Proposition 3.2. *If V is a compact manifold with boundary of dimension n, which is k-parallellizeable, then*

$$s\chi(\partial V) \equiv \chi(V) \qquad \textit{modulo } 2$$

for n not divisible by $2^{\phi(0,k)+1}$.

The proof of this result will be postponed until the end of this section.

Notes. a) For $k = 0$ this is classical. Specifically, if $M^{2j} = \partial V^{2j+1}$, then $\chi(M) = 2\chi(V)$.

b) For $k = \infty$ (or $k \geq n$) this is Kervaire's result [6] about the semi-characteristic of the boundary of framed manifold.

c) For V closed, this says $\chi(V)$ is even and is Theorem 2 of [15] applied with the results of [14].

The portion of this which is of interest for contact manifolds is the case $k = 1$, which may be phrased

Corollary 3.3. *If M is a closed oriented manifold of dimension $4k + 1$, and M is the boundary of the compact oriented manifold with boundary V, then the semi-characteristic of M is the mod 2 reduction of the Euler characteristic of V.*

Note. In [13] Reinhart defined an invariant for an oriented boundary M^{4k+1} given by the mod 2 reduction of $\chi(V)$ for any oriented V with $\partial V = M$. By the above, it is just the semi-characteristic of M.

Following Gray [4, § 3] one defines an *almost contact* manifold to be a manifold of dimension $2n + 1$ and an equivalence class of reductions of the tangent bundle to $1 \times U(n)$. If $P \to M$ is the bundle of frames of M with structure group $GL(2n + 1, R)$, an equivalence class of reductions to $1 \times U(n)$ is a homotopy class of cross-sections of the associated bundle with fiber $GL(2n + 1, R)/1 \times U(n)$.

From the last section, one sees that a strict contact manifold is an almost contact manifold, since there is a unique equivalence class of reductions to $O(2n+1)$, and the strict contact structure specifies the further reduction to $1 \times U(n)$ uniquely. It should be noted that the terminology is terrible: a contact manifold is *not* an almost contact manifold.

Proposition 3.4. *A closed almost contact manifold M^{2n+1} is the boundary of a compact stably almost complex manifold.*

Proof. A reduction of the tangent bundle of M to $1 \times U(n)$ gives a complex structure on the stable tangent bundle. Since the complex cobordism group Ω^U_{2n+1} is zero (Milnor [11]), M bounds as stably almost complex manifold.

Note. In essence this is contained in the proof of Theorem 2.3.2 of Gray [4].

This is the strongest plausible assertion, and has the easiest proof. Since a stably almost complex manifold is oriented, for example, it follows that M is also the boundary of an oriented manifold. Thus Corollary 3.3 applies to strict contact manifolds or almost contact manifolds of dimension $4k+1$.

The remainder of this section will be devoted to the proof of Proposition 3.2. Throughout, V will denote a k-parallellizeable compact n-manifold with boundary, and ∂V will denote its boundary.

If n is odd, one lets W be the manifold obtained from two copies of V by identifying boundaries. Then W is closed and odd dimensional so $\chi(W) = 0$, but $\chi(W) = \chi(V) + \chi(V) - \chi(\partial V)$, so $\chi(\partial V) = 2\chi(V)$.

Henceforth, one may then suppose n is even. For $n = 0$ there is nothing to prove, so let $n = 2j + 2$.

Lemma 3.5. *$s\chi(\partial V) = \chi(V) + \dim$ (image ϕ), where $\phi: H^{j+1}(V, \partial V; Z_2) \to H^{j+1}(V; Z_2)$.*

Proof. One has the exact cohomology sequence

$$0 \longrightarrow H^0(V, \partial V) \xrightarrow{\phi} H^0(V) \longrightarrow H^0(\partial V) \xrightarrow{\sigma} H^1(V, \partial V) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^j(V, \partial V) \longrightarrow H^j(V) \longrightarrow H^j(\partial V)$$

$$\xrightarrow{\sigma} H^{j+1}(V, \partial V) \xrightarrow{\sigma} \operatorname{im} \phi \longrightarrow 0$$

and the usual rule for Euler characteristics in an exact sequence gives

$$\sum_{i=0}^{j} (-1)^i \dim H^i(V) + (-1)^{j+1} \dim \operatorname{im} \phi$$

$$= \sum_{i=0}^{j+1} (-1)^i \dim H^i(V, \partial V) + \sum_{i=0}^{j} (-1)^i \dim H^i(\partial V) .$$

Since $\dim H^i(V, \partial V) = \dim H^{2j+2-i}(V)$ by Lefschetz duality, this equation gives $\chi(V) + \dim$ (image ϕ) $\equiv s\chi(\partial V)$. q.e.d.

Being given $a, b \in$ image ϕ, where $\phi\colon H^{j+1}(V, \partial V) \to H^{j+1}(V)$, one may choose $a', b' \in H^{j+1}(V, \partial V)$ with $\phi(a') = a$, $\phi(b') = b$. Then the value

$$\langle a' \cup b', [V, \partial V]\rangle$$

of the cup product of a' and b' on the fundamental class of $(V, \partial V)$ is independent of the choice of a' and b'. To see this, recall that $H^*(V, \partial V)$ is an $H^*(V)$ module and that there is a factorization of the cup product

$$\begin{array}{ccc} H^p(V, \partial V) \otimes H^q(V, \partial V) & \xrightarrow{\ \cup\ } & H^{p+q}(V, \partial V) \\ \downarrow{\scriptstyle \phi \otimes 1} & \nearrow{\scriptstyle \cup} & \\ H^p(V) \otimes H^q(V, \partial V) & & \end{array}$$

so that if $\phi(a') = \phi(a'') = a$ and $\phi(b') = \phi(b'') = b$, then $\langle a' \cup b', [V, \partial V]\rangle = \langle \phi(a') \cup b', [V, \partial V]\rangle = \langle \phi(a'') \cup b', [V, \partial V]\rangle = \langle a'' \cup b', [V, \partial V]\rangle = \langle a'' \cup \phi(b'), [V, \partial V]\rangle = \langle a'' \cup \phi(b''), [V, \partial V]\rangle = \langle a'' \cup b'', [V, \partial V]\rangle$.

Thus one has a symmetric bilinear form

$$[\ ,\ ]\colon (\text{image } \phi) \otimes (\text{image } \phi) \to Z_2$$

defined by $[a, b] = \langle a' \cup b', [V, \partial V]\rangle$ where $\phi(a') = a, \phi(b') = b$. This pairing is nondegenerate. To see this, note that if $[a, b] = 0$ for all b, then letting $a = \phi(a')$ one has, for every $b' \in H^{j+1}(V, \partial V)$, $\langle \phi(a') \cup b', [V, \partial V]\rangle = \langle a' \cup b', [V, \partial V]\rangle = [a, \phi(b')] = 0$, and $a = \phi(a') = 0$ by Lefschetz duality.

From the theory of nondegenerate symmetric bilinear forms over Z_2, there is a unique class $v \in$ image ϕ for which $[v, a] = [a, a]$ for all a, and the dimension of image ϕ is congruent modulo 2 to the value $[v, v]$. Since $[a, a] = \langle a' \cup a', [V, \partial V]\rangle = \langle Sq^{j+1}a', [V, \partial V] = \langle v_{j+1} \cup a', [V, \partial V]\rangle$, where $v_{j+1} \in H^{j+1}(V)$ is the $(j + 1)$-st Wu class of $(V, \partial V)$, and since v_{j+1} restricts to the $(j + 1)$-st Wu class of ∂V, which is zero since $Sq^{j+1}\colon H^j(\partial V) \to H^{2j+1}(\partial V)$ is zero, $v_{j+1} \in$ image ϕ, giving $[a, a] = [v_{j+1}, a]$. Thus v is the Wu class v_{j+1}.

Now since V is k-parallellizeable, the tangent map $\tau\colon V \to BO$ lifts to the connective cover $BO(k + 1, \cdots, \infty)$ of BO. By [14], the Stiefel-Whitney classes $w_i(V)$ of V are zero if $i < 2^{\phi(0,k)}$, and so the Wu classes $v_i(V)$ are zero if $i < 2^{\phi(0,k)}$, so $Sq^i\colon H^{2j+2-i}(V, \partial V) \to H^{2j+2}(V, \partial V)$ is zero if $i < 2^{\phi(0,k)}$. By Proposition 6 of [15], $Sq^i \in \sum_{t=0}^{s} Sq^{2^t}\mathscr{A}$ if $i \not\equiv 0\ (2^{s+1})$, so $Sq^i\colon H^{2j+2-i}(V, \partial V) \to H^{2j+2}(V, \partial V)$ is zero if $i \not\equiv 0\ (2^{\phi(0,k)})$. In particular, if $n \not\equiv 0\ (2^{\phi(0,k)+1})$, $j + 1 \not\equiv 0\ (2^{\phi(0,k)})$, so $v_{j+1} = 0$ and dim (image ϕ) $\equiv [v_{j+1}, v_{j+1}] \equiv 0$. From the Lemma, $s\chi(\partial V) \equiv \chi(V)$.

Note. The fact that the image of $\phi\colon H^*(V, \partial V; Z_2) \to H^*(V; Z_2)$ satisfies "Poincaré" duality appears in the dissertation of one of the author's students, Russell J. Rowlett, and in rational cohomology for oriented manifolds was used by Atiyah and Singer.

4. Cobordism theoretic approach

It has been shown by Ocken [12] that Reinhart's cobordism group [13] of n-manifolds (or oriented n-manifolds) is given by cobordism of manifolds with a reduction of the stable tangent bundle to $O(n)$ (or $SO(n)$). This places Reinhart's groups and invariants in a broader theoretical context.

In order to make similar use of cobordism theory in studying contact manifolds, one seeks an appropriate class of stable tangent structure possessed by strict contact manifolds but stronger than stable almost complex structure, so that a strict contact manifold will not bound. There is a rather obvious candidate.

Definition 4.1. A compact manifold (with or without boundary) of dimension $2n+1$ or $2n+2$ will be called a *stably almost contact manifold* if it is given an equivalence class of reductions of the stable tangent bundle to $U(n)$.

This is an appropriate class of manifolds with which to define a cobordism group. Specifically, an element of this group is an equivalence class of closed stably almost contact manifolds of dimension $2n+1$. The zero class is the class of boundaries of stably almost contact manifolds of dimension $2n+2$, and the group operation is induced by disjoint union. This group will be denoted $\Omega^{U(n)}_{2n+1}$.

The formalism needed to make this precise may be found in Lashof [8]. Specifically, a stably almost contact manifold of dimension $2n+1$ or $2n+2$ is a manifold with (B, f) structure in the sense of Lashof, where B is the classifying space $BU(n)$ for complex n-plane bundles and $f\colon BU(n) \to BO$ is a map classifying the complement of the universal complex n-plane bundle. The group $\Omega^{U(n)}_{2n+1}$ is the $(2n+1)$-dimensional cobordism group of such (B, f) manifolds.

The main result of this section is:

Proposition 4.2. *The group $\Omega^{U(n)}_{2n+1}$ is 0 if n is odd, and is Z_2 if n is even.*

Being given a closed oriented contact manifold M of dimension $4k+1$, one lets $J(M)$ denote the class of M in $Z_2 = \Omega^{U(2k)}_{4k+1}$. $J(M)$ will also be used to denote the class of an almost contact or stably almost contact manifold.

It will be shown that $J(M)$ coincides with the semi-characteristic for almost contact manifolds.

In order to compute $\Omega^{U(n)}_{2n+1}$, one may apply the generalized Pontrjagin-Thom theorem from Lashof [8]. Being given the map $f\colon BU(n) \to BO$ one forms an associated Thom spectrum $MU\langle n\rangle$ and has

$$\Omega^{U(n)}_{2n+1} \cong \Pi^S_{2n+1}(MU\langle n\rangle)$$

given by the stable homotopy group of the spectrum.

Letting $g\colon BU(n) \to BU$ classify the universal bundle and $h\colon BU \to BO$ classify the complement of the universal bundle, one has $f = h \circ g$, and has induced a map of spectra $Tg\colon MU\langle n\rangle \to MU$. The stable homotopy of MU gives the complex cobordism ring $\Omega^U_j \cong \pi^S_j(MU)$, and the homotopy homomorphism induced by Tg is the forgetful homomorphism

$$\Omega^{U(n)}_{2n+1} \to \Omega^U_{2n+1}$$

which considers a stably almost contact manifold as stably almost complex.

The homotopy exact sequence of the map (pair) $Tg\colon MU\langle n\rangle \to MU$ gives an exact sequence

$$\cdots \longrightarrow \Omega^U_{2n+2} \xrightarrow{i} \pi^S_{2n+2}(MU, MU\langle n\rangle) \xrightarrow{\partial} \Omega^{U(n)}_{2n+1} \longrightarrow \Omega^U_{2n+1} = 0 .$$

The relative homotopy group $\pi^S_{2n+1}(MU, MU\langle n\rangle)$ may be interpreted geometrically as the relative cobordism group [16, p. 25] formed from stably almost complex manifolds with boundary of dimension $2n + 2$ with a compatible stably almost contact structure on the boundary. The homomorphism ∂ takes the class of the boundary manifold, while i is obtained by considering a closed stably almost complex manifold as a manifold with boundary whose boundary happens to be empty.

In order to compute $\pi^S_{2n+2}(MU, MU\langle n\rangle)$, one notes that $g\colon BU(n) \to BU$ induces an epimorphism

$$\begin{array}{ccc} g^*: H^*(BU(n); Z) & \leftarrow & H^*(BU; Z) \\ \| & & \| \\ Z[c_i \mid i \leq n] & & Z[c_i] \end{array}$$

with $g^*(c_i) = c_i$ for $i \leq n$, $g^*(c_i) = 0$ for $i > n$, c_i being the i-th universal Chern class. Hence $H^*(BU, BU(n); Z)$ is the ideal in $H^*(BU; Z)$ generated by the $c_j, j > n$. Applying the Thom isomorphism $H^*(MU, MU\langle n\rangle; Z) \cong H^*(BU, BU(n); Z)$ so that $H_i(MU, MU\langle n\rangle; Z) = 0$ for $i < 2n + 2$ and is Z for $i = 2n + 2$. By the Hurewicz theorem, $\pi^S_i(MU, MU\langle n\rangle)$ is 0 for $i < 2n + 2$ and is Z for $i = 2n + 2$.

One may then study the composite

$$\begin{array}{c} \Omega^U_{2n+2} \xrightarrow{i} \pi^S_{2n+2}(MU, MU\langle n\rangle) \xrightarrow{H} H_{2n+2}(MU, MU\langle n\rangle; Z) \\ \wr\| \\ H_{2n+2}(BU, BU\langle n\rangle; Z) = Z \end{array}$$

where $H_{2n+2}(BU, BU\langle n\rangle; Z)$ is identified with the integers by assigning to a homology class a the value of the Kronecker pairing $\langle c_{n+1}, a\rangle$. For a manifold pair $(V, \partial V)$ of $\pi^S_{2n+2}(MU, MU\langle n\rangle)$, the tangent map $\tau\colon (V, \partial V) \to (BU, BU(n))$ induces a relative Chern class $c_{n+1}(\tau) \in H^{2n+2}(V, \partial V; Z)$, and the integer assigned to $(V, \partial V)$ is $\langle c_{n+1}(\tau), [V, \partial V]\rangle$. In particular, if V is a closed stably almost complex manifold, this is the value of the top Chern number $c_{n+1}[V]$.

Among all closed stably almost complex manifolds V of dimension $2n + 2$, the integers arising as the values of $c_{n+1}[V]$ are all integers if n is odd or all even integers if n is even. To see this $CP(1)^{n+1}$ has value 2^{n+1} and $CP(n+1)$ has value $n+2$, so that if n is odd, the greatest common divisor is one. If n is even, $CP(1) \times CP(n)$ has value $2(n+1)$, so that the greatest common divisor is 2. Since $c_{n+1}[V]$ mod 2 is the Stiefel-Whitney number $w_{2n+2}[V]$ or the Euler characteristic mod 2, and since a closed oriented manifold of dimension not divisible by 4 has even Euler characteristic, $c_{n+1}[V]$ is always even if n is even.

Thus the homomorphism $i\colon \Omega^U_{2n+2} \to \pi^S_{2n+2}(MU, MU\langle n\rangle) = Z$ is epic for n odd and maps onto $2Z$ for n even. From the homotopy exact sequence

$$\Omega^{U\langle n\rangle}_{2n+1} = \begin{cases} 0 & n \text{ odd}\ , \\ Z_2 & n \text{ even}\ , \end{cases}$$

giving Proposition 4.2.

Being given a closed stably almost contact manifold M of dimension $4k+1$, the proof just given shows how to compute $J(M) \in Z_2$. Specifically, one chooses any stably almost complex manifold V whose boundary is M and $J(M)$ is the mod 2 reduction of $c_{2k+1}(\tau)[V, M]$. Unfortunately, finding such a V for which the structure restricts properly and carrying through the evaluation are not practical.

In order to simplify the calculation proceedure, one may consider the diagram of classifying spaces

$$\begin{array}{ccccc} BU(2k) & \xrightarrow{g} & BU & \xrightarrow{h} & BO \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle 1} \\ BSO(4k+1) & \xrightarrow{p} & BSO & \xrightarrow{q} & BO \end{array}$$

where r, s, p classify the universal bundles, and q classifies the complement. For each of these one has a Thom spectrum and induced obvious maps. In particular, one has (s, r) inducing a homomorphism

$$\pi^S_{4k+1}(MU, MU\langle 2k\rangle) \to \pi^S_{4k+2}(MSO, MSO\langle 4k+1\rangle)\ ,$$

which will be studied.

To begin, consider the homomorphism

$$\begin{array}{ccc} p^*\colon H^*(BSO(4k+1)\,;Z_p) & \leftarrow & H^*(BSO\,;Z_p) \\ \| & & \| \\ Z_p[\mathscr{P}_i \mid i \le 2k] & & Z_p[\mathscr{P}_i] \end{array}$$

where p is an odd prime and $\mathscr{P}_i$ is the i-th Pontrjagin class. p^* is an isomorphism in dimensions less than $8k+4$, so that $H^*(MSO, MSO\langle 4k+1\rangle\,; Z_p) \cong H^*(BSO, BSO(4k+1)\,; Z_p)$ is zero below dimension $8k+4$.

Also,

$$\begin{array}{ccc} p^*: H^*(BSO(4k+1); Z_2) & \longleftarrow & H^*(BSO; Z_2) \\ \| & & \| \\ Z_2[w_i \mid 1 < i \leq 4k+1] & & Z_2[w_i \mid 1 < i] \end{array}$$

is epic with kernel the ideal generated by the $w_i, i > 4k+1$, so that $H^i(MSO, MSO\langle 4k+1\rangle; Z_2) \cong H^i(BSO, BSO(4k+1); Z_2)$ is zero for $i < 4k+2$ and is Z_2 for $i = 4k+2$. Since the bundles are oriented, the Thom isomorphism commutes with the Steenrod operation Sq^1. Since $Sq^1 w_{4k+2} = w_{4k+3}$ in BSO, Sq^1 is nonzero on $H^{4k+2}(MSO, MSO\langle 4k+1\rangle; Z_2)$.

Combining these, one has $H_i(MSO, MSO\langle 4k+1\rangle; Z)$ equal to zero for $i < 4k+2$ and to Z_2 for $i = 4k+2$, and by the Hurewicz theorem $\pi^S_{4k+2}(MSO, MSO\langle 4k+1\rangle)$ is Z_2 and maps isomorphically to the homology.

In mod 2 cohomology, the homomorphism

$$H^{4k+2}(MSO, MSO\langle 4k+1\rangle; Z_2) \to H^{4k+2}(MU, MU\langle 2k\rangle; Z_2)$$

induced by (s, r) is an isomorphism, so that the induced homotopy homomorphism is epic. Thus one has the commutative diagram

$$\begin{array}{ccc} \pi^S_{4k+2}(MU, MU\langle 2k\rangle) & \longrightarrow & \pi^S_{4k+2}(MSO, MSO\langle 4k+1\rangle) \\ \downarrow\cong & & \downarrow\cong \\ H_{4k+2}(MU, MU\langle 2k\rangle; Z) & \to & H_{4k+2}(MSO, MSO\langle 4k+1\rangle; Z) \\ \wr\| & & \wr\| \\ Z & \xrightarrow[\text{epic}]{} & Z_2 \end{array}$$

inducing isomorphisms

$$\begin{aligned} \Omega^{U(2k)}_{4k+1} &\cong \pi^S_{4k+2}(MU, MU\langle 2k\rangle)/2\pi^S_{4k+2}(MU, MU\langle 2k\rangle) \\ &\cong \pi^S_{4k+2}(MSO, MSO\langle 4k+1\rangle) . \end{aligned}$$

Interpreting the relative homotopy group as a relative cobordism group, one may now describe $J(M)$ more reasonably. Being given the closed stably almost contact manifold M of dimension $4k+1$ one chooses any *oriented* manifold V whose boundary is M, thus (V, M) represents $J(M)$ in $\pi^S_{4k+2}(MSO, MSO\langle 4k+1\rangle)$. The tangent map is given by a map of pairs $\tau: (V, M) \to (BSO, BSO(4k+1))$ giving $w_{4k+2}(\tau) \in H^{4k+2}(V, M; Z_2)$, and $J(M) \in Z_2$ is the value of $\langle w_{4k+2}(\tau), [V, M]\rangle$.

It should be noted that the $SO(4k+1)$ structure on an almost contact manifold M is the obvious one, namely, the $SO(4k+1)$ structure arising from the stabilization of the tangent bundle of M, and that a general element of $\pi^S_{4k+2}(MSO, MSO\langle 4k+1\rangle)$ or a class obtained from a stably almost contact manifold is represented by a V and some reduction along ∂V, not necessarily the obvious reduction to the tangent bundle of ∂V.

It is clear for an almost contact manifold that the tangent map may be considered as a map sending V into $BSO(4k+2)$ since the tangent bundle of V is a $(4k+2)$-plane bundle, and $\langle w_{4k+2}(\tau), [V, M]\rangle$ is the mod 2 reduction of $\langle X(\tau), [V, M]\rangle$ where $X(\tau)$ is obtained by pulling back the Euler class in $H^{4k+2}(BSO(4k+2), BSO(4k+1); Z)$. The value of $\langle X(\tau), [V, M]\rangle$ is $\pm\chi(V)$ where χ is the Euler characteristic, with the sign depending on the use of inward or outward pointing normal field to orient V and M compatibly. (See Gramain [3] for example.) This gives the result promised.

Proposition 4.3. *If M^{4k+1} is a closed almost contact manifold, then the class of M in $\Omega_{4k+1}^{U(2k)} = Z_2$, denoted $J(M)$, is given by the semi-charateristic of M. In particular, the semi-characteristic measures the failure of M to bound a stably almost contact manifold.*

Note. The invariant $J(M)$ for an almost contact manifold M presupposes that M is given the induced stably almost contact structure. A manifold M may possess two stably almost contact structures, for which the classes in $\Omega_{4k+1}^{U(2k)}$ are distinct. For example the circle S^1 has two distinct framings of its stable tangent bundle, and these give the two classes in $\Omega_1^{U(0)}$; only one of these stably almost contact structures comes from the contact structure on S^1.

5. Boundaries of almost complex manifolds

Being given a compact almost complex manifold with boundary V of dimension $2n+2$, the tangent bundle of V has an operator J covering the identity map and satisfying $J^2 = -1$. The restriction of the tangent bundle of V to $M = \partial V$ has one section given by the unit inward pointing normal, the orthogonal complement being the tangent bundle to M. Applying J to the normal field gives a section of the tangent bundle of M, and the complement is a complex bundle. Thus the boundary of V has an almost contact structure induced by the almost complex structure on V.

The classical examples of contact manifolds all arise in this way. One would then like to improve Proposition 3.4 by eliminating the word "stably". There is one case in which this can be done, and one has

Proposition 5.1. *A closed almost contact manifold M of dimension $8k+5$ is the boundary of a compact almost complex manifold.*

Proof. Let M be a closed almost contact manifold of dimension $8k+5$. If the result is true for each component of M, then it holds for M, and without loss of generality one may suppose M to be connected.

Applying Proposition 3.4, one has $M = \partial V$ for some compact stably almost complex manifold V, and by taking the connected component of V containing M one may assume V to be connected.

The tangent map $\tau: (V, M) \to (BU, BU(4k+2))$ may be deformed into

$BU(4k+3)$ keeping the map fixed on M since V has dimension $8k+6$ and $\pi_i(BU, BU(4k+3)) = 0$ for $i \leq 8k+6$. Thus there are a complex $(4k+3)$-bundle η over V and an isomorphism of η/M with $\tau_M \oplus 1$ so that η induces the almost contact structure of M and that the stabilization of η gives the stably almost complex structure of V, i.e., one has a chosen stable isomorphism of η and τ_v.

If one just had an isomorphism of η and τ_v compatible with the given isomorphism over M, then V would be the desired almost complex manifold with the structure arising from the isomorphism of τ_v and η. Thus one considers the maps $\tau_v, \eta\colon (V, M) \to (BSO(8k+6), BU(4k+2))$ and seeks a homotopy joining them and keeping M fixed. Since τ_v and η are stably isomorphic, their restrictions to the $(8k+5)$-skeleton of V are isomorphic, and thus one may assume that the maps agree on this skeleton. The obstruction to finding the desired homotopy is then the difference element

$$d(\tau_v, \eta) \in H^{8k+6}(V, M; \pi_{8k+6}(BSO(8k+6))) .$$

From the transgression homomorphism

$$\Delta : \pi_{8k+6}(BSO(8k+6)) \to \pi_{8k+5}(S^{8k+5}) = Z$$

of the fibering $S^{8k+5} \to BSO(8k+5) \xrightarrow{\pi} BSO(8k+6)$ one has induced a coefficient homomorphism

$$\Delta_{\#}\colon H^{8k+6}(V, M; \pi_{8k+6}(BSO(8k+6))) \to H^{8k+6}(V, M; \pi_{8k+5}(S^{8k+5})) ,$$

and

$$\Delta_{\#} d(\tau_v, \eta) = X(\tau_v) - X(\eta)$$

is the difference of the relative Euler classes of τ_v and η (the obstructions to lifting in π with the given lift on the boundary M). See James and Thomas [5, pp. 500–501].

Since V and M are connected, $H^{8k+6}(V, M; \mathscr{R}) \cong \mathscr{R}$ and $\Delta_{\#}$ is identified with the homomorphism Δ. From Kervaire [7], one has

$$\begin{array}{ccccc}
\pi_{8k+6}(BSO(8k+5)) & \longrightarrow & \pi_{8k+6}(BSO(8k+6)) & \xrightarrow{\Delta} & \pi_{8k+5}(S^{8k+5}) \\
\wr\| & & \wr\| & & \| \\
\pi_{8k+5}(SO(8k+5)) & \longrightarrow & \pi_{8k+5}(SO(8k+6)) & & Z \\
\| & & \| & & \\
Z_2 & & Z & &
\end{array}$$

so that Δ is monic. Further, since V and M are connected one has $X(\tau_v) - X(\eta) = 0$ if and only if the value of the Kronecker product $\langle X(\tau_v) - X(\eta), [V, M]\rangle$ is zero.

Summarizing, one must find a V for which $\langle X(\tau_v), [V, M]\rangle = \pm\chi(V)$ (the sign depending on the orientation convention) is equal to $\langle X(\eta), [V, M]\rangle$. Since the relative Euler class of a complex $(4k+3)$-bundle is its Chern class c_{4k+3}, $X(\eta) = c_{4k+3}(\eta)$, and the relative Chern class of η is the same as that of its stabilization $(H^{8k+6}(BU, BU(4k+2); Z) \cong H^{8k+6}(BU(4k+3), BU(4k+2); Z))$, one sees that it suffices to find a connected stably almost complex manifold V with $\partial V = M$ for which

$$\langle c_{4k+3}(\tau), [V, M]\rangle = \pm\chi(V) ,$$

where the Kronecker product is as discussed in § 4, and is the determining invariant of $\pi^S_{8k+6}(MU, MU\langle 4k+2\rangle)$.

To begin finding such a V, one first chooses any connected V' with $\partial V' = M$ and modifies it. First, being given V' and any connected closed stably almost complex manifold W, one may form the connected sum $V'\# W$ by doing surgery on S^0 imbedded in $V' \cup W$ with one point in the interior of V' and one point in W. The resulting manifold $V'\# W$ is connected. As in Lashof [8], this surgery may be accomplished as a $(BU, BU(4k+2))$ cobordism without changing the boundary structure.

Since the relative Chern number is a cobordism invariant,

$$\langle c_{4k+3}(\tau), [V'\# W, M]\rangle = \langle c_{4k+3}(\tau), [V', M]\rangle + \langle c_{4k+3}(\tau), [W]\rangle ,$$

while the Euler characteristics are related by

$$\chi(V'\# W) = \chi(V') + \chi(W) - 2 .$$

(See Lemma 1 of Reinhart [13]; one is replacing $D^{8k+6} \times S^0$ by $S^{8k+5} \times D^1$, i.e., it is a surgery of type $(8k+5, 0)$).

In particular, if $W = S^1 \times S^{8k+5}$ with the trivial stably almost complex structure for which it bounds, this does not change the Chern number $(\langle c_{4k+3}(\tau), [W]\rangle = 0)$ but subtracts 2 from the Euler characteristic. If instead one lets $W = S^2 \times S^{8k+4}$ with the trivial stably almost complex structure, the Chern number is unchanged but 2 is added to the Euler characteristic. Since when reduced modulo 2 the Chern and Euler number agree (as in § 4), it follows that by iterating this process one may find a V with the desired Chern number-Euler number relationship. Thus one may find the desired almost complex manifold V, completing the proof.

Note. Dimensions of the form $8k+5$ are the only ones for which Δ is monic; that is the only point at which the hypothesis was used.

One may also apply this to understand $s\chi(M)$ more thoroughly. Let M be a closed connected almost contact manifold of dimension $8k+5$ for which $s\chi(M) = 0$. One may then choose a stably almost complex manifold V with $\partial V = M$ for which $\langle c_{4k+3}(\tau), [V, M]\rangle = 0$, and by surgery one may form the connected sum of the components of V, i.e., may assume V to be connected. By attaching copies of $S^1 \times S^{8k+5}$ and $S^2 \times S^{8k+4}$, one may also obtain $\chi(V) = 0$ so that V is an almost complex manifold whose tangent bundle has a section extending the inward pointing normal field along $\partial V = M$. Multiplying this field by J gives a second field and reduces the tangent bundle of V to $1 \times U(4k+2)$ compatibly with the reduction on M. This proves

Proposition 5.2. *A closed almost contact manifold M of dimension $8k+5$ is the boundary of a compact manifold with tangent bundle admitting a reduction to $1 \times U(4k+2)$ compatible with the reduction on M if and only if $s\chi(M) = 0$.*

6. Characteristic classes

The object of this section is to describe the characteristic classes of a contact manifold, or more precisely, the cohomology of the classifying space BC_{2n+1}.

Before beginning the calculations, notice that if $0(2n+1)$ is included in $0(2n+2)$, then C_{2n+1} becomes the subgroup of $0(2n+2)$ generated by $1 \times U(n)$ and conjugation (where $R^{2n+2} = C^{n+1}$). Thus C_{2n+1} is the intersection of $0(2n+1)$ and $\tilde{U}(n+1)$ in $0(2n+2)$, where $\tilde{U}(n+1)$ is the group of real linear transformations $T: C^{n+1} \to C^{n+1}$ which preserve the real inner product and are either complex linear or conjugate linear (i.e., $Tix = iTx$ for all x or $Tix = -iTx$ for all x).

Alternately, C_{2n+1} may be identified with $\tilde{U}(n)$ (although not as usually contained in $0(2n) \subset 0(2n+1)$). To see this, one has a homomorphism $\phi: \tilde{U}(n) \to Z_2 = 0(1)$ with kernel $U(n)$, and letting $i: \tilde{U}(n) \to 0(2n)$ be the inclusion one obtains that C_{2n+1} is the image of

$$\tilde{U}(n) \xrightarrow{\phi \times i} 0(1) \times 0(2n) \hookrightarrow 0(2n+1) .$$

The groups $\tilde{U}(n)$ have been extensively studied by one of the author's students, Mr. Paul Beem, and the remarks which follow are contained in his dissertation.

First notice that from the exact sequence $1 \to 1 \times U(n) \to C_{2n+1} \to Z_2 \to 1$ there is a double cover of BC_{2n+1} by $BU(n)$. The covering transformation is the involution induced by conjugation on $U(n)$. Thus the projection $\pi: BU(n) \to BC_{2n+1}$ induces a monomorphism $\pi^*: H^*(BC_{2n+1}; \mathscr{R}) \to H^*(BU(n); \mathscr{R})$ for any ring $\mathscr{R}$ containing $\frac{1}{2}$, and the image of π^* consists of those elements invariant under the involution. The ring $H^*(BU(n); \mathscr{R})$ is the polynomial ring over $\mathscr{R}$ on the Chern classes $c_i \in H^{2i}(BU(n); \mathscr{R})$, and the involution sends c_i to $(-1)^i c_i$, so that $H^*(BC_{2n+1}; \mathscr{R})$ is completely known.

With Z_2 coefficients, one uses the fibration $BU(n) \xrightarrow{j} BC_{2n+1} \xrightarrow{p} BZ_2$ coming from the exact sequence. Since the composite $BU(n) \xrightarrow{j} BC_{2n+1} \xrightarrow{i}$

$BO(2n+1)$ is epic in mod 2 cohomology, the fibration p is totally non-homologous to zero, and $H^*(BC_{2n+1}; Z_2)$ is the polynomial algebra over Z_2 on $p^*(w_1)$ and $i^*(w_{2k})$ for $1 \leq k \leq n$. If n is even, $C_{2n+1} \not\subset SO(2n+1)$ so $p^*(w_1) = i^*(w_1)$, while for n odd, $C_{2n+1} \subset SO(2n+1)$ and $i^*(w_1) = 0$. More generally, one wishes to express $i^*(w_{2k+1})$ in terms of $p^*(w_1)$ and the $i^*(w_{2k})$.

Being given a space Y and a principal C_{2n+1} bundle over $Y, \pi: P \to Y$, $P/1 \times U(n) = X$ is a double cover of Y induced by the homomorphism $C_{2n+1} \to Z_2 = 0(1)$ and giving a line bundle ξ over Y (the contact line bundle). The inclusion $C_{2n+1} \to 0(1) \times 0(2n)$ also provides a principal $0(2n)$ bundle over Y with associated real $2n$-plane bundle η over Y (the contact distribution) and further inclusion in $0(2n+1)$ gives the associated $(2n+1)$-plane bundle $\tau = \xi \oplus \eta$ (the tangent bundle or cotangent bundle for contact manifold).

If ρ denotes the vector bundle over X induced from η, then ρ is a complex n-plane bundle. Letting $T: X \to X$ be the involution given by the interchange of sheets in the double cover, one has a covering involution T^* on ρ which is a conjugation ($T^*i = -iT^*$). One may consider Y as the orbit space X/T, ξ as the line bundle with total space $X \times R/(x, r) \sim (Tx, -r)$, and η as the $2n$-plane bundle with total space $E(\eta) = E(\rho)/T^*$.

Claim. $\eta \otimes \xi \cong \eta$.

Proof. For $e \in E(\rho)$, let $\pi_\rho(e)$ be the projection of e in X. Let $[e]$ denote the class of e in $E(\rho)/T^* = E(\eta)$, and let $\{x, r\}$ denote the class of (x, r) in $E(\xi) = X \times R/\sim$. Consider the function $\theta: E(\rho) \to E(\eta \otimes \xi)$ given by $\theta(e) = [ie] \otimes \{\pi_\rho(e), 1\}$. This covers the projection of X on Y and is real-linear on fibers. Now

$$\begin{aligned}\theta(T^*e) &= [iT^*e] \otimes \{\pi_\rho(T^*e), 1\} = [-T^*ie] \otimes \{T\pi_\rho(e), 1\} \\ &= [-ie] \otimes \{\pi_\rho(e), -1\} = [ie] \otimes \{\pi_\rho(e), 1\} = \theta(e)\ ,\end{aligned}$$

so that θ induces a map of $E(\rho)/T^* = E(\eta)$ into $E(\eta \otimes \xi)$ which is obviously a bundle isomorphism.

From the equation $\eta \otimes \xi = \eta$, one has

$$(\tau \otimes \xi) \oplus \xi = (\eta \oplus \xi) \otimes \xi = (\eta \otimes \xi) \oplus 1 \oplus \xi = \tau \oplus 1\ .$$

Letting $c \in H^1(Y; Z_2)$ be $w_1(\xi)$ and letting $w_i \in H^i(Y : Z_2)$ be the i-th Stiefel-Whitney class of $\tau, i \leq 2n+1$, one has

$$\begin{aligned}1 + w_1 + w_2 + \cdots + w_{2n+1} &= w(\tau \oplus 1) = w((\tau \otimes \xi) \oplus \xi) \\ &= \{(1+c)^{2n+1} + (1+c)^{2n} w_1 + \cdots + w_{2n+1}\}(1+c) \\ &= (1+c)^{2n+2} + (1+c)^{2n+1} w_1 + \cdots \\ &\quad + (1+c)^{2n+2-r} w_r + \cdots + (1+c) w_{2n+1}\ .\end{aligned}$$

Now specialize to $Y = BC_{2n+1}$, so that $c = p^*(w_1)$ and $w_i = i^*(w_i)$ in the previous notation (so that multiplication by c is monic). The above equation then becomes

$$\sum_{r=0}^{2n+1} \frac{1}{c}[(1+c)^{2n+2-r}+1]w_r = 0 ,$$

and $[(1+c)^{2n+2-r}+1]/c = \binom{2n+2-r}{1}$ + terms in c, so that the odd degree components of this equation give polynomial identities for w_{2k+1} in terms of c and the $w_i, i < 2k+1$.

Claim.

$$\begin{aligned} 1 + w_1 + w_2 + \cdots + w_{2n+1} &= c^{-1}[(1+c^2)^{n+1}+(1+c)] + c^{-1}[(1+c^2)^n+(1+c)]w_2 + \cdots \\ &\quad + c^{-1}[(1+c^2)^{n+1-r}+(1+c)]w_{2r} + \cdots \\ &\quad + c^{-1}[(1+c^2)+(1+c)]w_{2n} . \end{aligned}$$

Proof. At first one notes that

$$[(1+c^2)^j + (1+c)]/c = 1 + \text{odd powers of } c ,$$

so that the even degree terms are just the w_{2k} on both sides, and the odd degree terms give identities for w_{2k+1} as polynomials in c and the w_{2j}.

It is sufficient to show that

$$A = \sum_{r=0}^{n} \frac{1}{c}[(1+c^2)^{n+1-r}+(1+c)]w_{2r}$$

satisfies the identity given by $w(\tau \oplus 1) = w((\tau \otimes \xi) \oplus \xi)$ since this identity implies unique formulas for the w_{odd}. Thus it suffices to show that A is transformed into itself if the homogeneous component degree i is multiplied by $(1+c)^{-i}$ and the entire expression is then multiplied by $(1+c)^{2n+2}$. Under this operation, $\{[(1+c^2)^{n+1-r}+(1+c)]/c\}w_{2r}$ is transformed into

$$\begin{aligned} &(1+c)^{2n+2}\left\{\frac{[1+(c/(1+c))^2]^{n+1-r}+[1+c/(1+c)]}{c/(1+c)}\right\}\frac{w_{2r}}{(1+c)^{2r}} \\ &\quad = (1+c)^{2n+3-2r}c^{-1}[(1+c)^{-2(n+1-r)}+(1+c)^{-1}]w_{2r} \end{aligned}$$

(using the identities $1+x^2=(1+x)^2$ and $1+c/(1+c)=(1+c)^{-1}$)

$$= c^{-1}[(1+c)+(1+c)^{2n+2-2r}]w_{2r} .$$

Thus A is unchanged, giving the result.

In conclusion, one has a complete description of the mod 2 cohomology of

BC_{2n+1}, and knows the relaltions between the Stiefel-Whitney classes and the Stiefel-Whitney class of the contact line bundle for a contact manifold.

Corollary. *A closed contact manifold M of dimension less than or equal to* 11 *is the boundary of a compact manifold with boundary.*

The proof is an uninspiring exercise in horrid calculation, and is not worth writing down. It would be pleasant if this were true without dimensional restrictions, but a case by case checking of Stiefel-Whitney numbers is no way to prove it.

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